Binomial Functions

Binomial Series

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad (|x|<1, n \in \mathbb{R})$$

Q1

Find the term in x^3 in the binomial expansion of

$$(1+2x)^{-1}$$ [4]

FORMULA BOOKLET

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad (|x|<1, n \in \mathbb{R})$$

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{1\times 2}(2x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(2x)^3 + \ldots$$

$$(1+x)^n$$

$$= 1 - 2x + 4x^2 - 8x^3 + \ldots$$

Q2

Find the first 3 terms in the binomial expansion of

$$\frac{1}{(3-x)^2}$$ [7]

Find the first 3 terms in the binomial expansion of

$$\frac{1}{(3-x)^2}$$

FORMULA BOOKLET

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \quad (|x|<1, n \in \mathbb{R})$$

$$\frac{1}{(3-x)^2} = (3-x)^{-2} = \left[3\left(1-\frac{x}{3}\right)\right]^{-2} = \frac{1}{9}\left[\left(1-\frac{x}{3}\right)^{-2}\right]$$

expand this out then multiply by 1/9

$$\left(1-\frac{x}{3}\right)^{-2} = 1 + (-2)\left(\frac{-x}{3}\right) + \frac{(-2)(-3)}{1\times 2}\left(-\frac{x}{3}\right)^2 + \frac{(-2)(-3)(-4)}{1\times 2\times 3}\left(-\frac{x}{3}\right)^3 + \ldots$$

$$= \underbrace{1 + \frac{2x}{3} + \frac{x^2}{3}}_{\text{only need first three terms}} + \frac{4x^3}{27} + \ldots$$

Now multiply by 1/9

$$\approx \frac{1}{9}\left[1 + \frac{2x}{3} + \frac{x^2}{3}\right]$$

$$\approx \frac{1}{9} + \frac{2x}{27} + \frac{x^2}{27}$$

Answer $\frac{1}{9} + \frac{2x}{27} + \frac{x^2}{27}$

Q3

Find the binomial expansion of

$$\sqrt{1-x^2} \qquad (1-x^2)^{\frac{1}{2}}$$

up to and including the term in x^4 [6]

$$(1-x^2)^{\frac{1}{2}} \approx 1 + \left(\tfrac{1}{2}\right)(-x^2) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{1\times 2}(-x^2)^2 + \ldots\ldots$$

no more terms required

$(1+x)^n$ formula booklet

$$\approx 1 - \frac{x^2}{2} - \frac{x^4}{8}$$

Answer $1 - \dfrac{x^2}{2} - \dfrac{x^4}{8}$

Q4

Use the binomial theorem to expand in ascending powers of x

$$\frac{1+x}{(1-x)^{\frac{1}{2}}}$$

up to and including the term in x^2 [7]

Use the binomial theorem to expand in ascending powers of x

$$\frac{1+x}{(1-x)^{\frac{1}{2}}} = (1+x)(1-x)^{-1/2}$$

up to and including the term in x^2

Firstly, expand out $(1-x)^{-1/2}$ using formula given in formula booklet

$$(1-x)^{-1/2} \approx 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times 2}(-x)^2 + \cdots$$

$$\approx 1 + \frac{x}{2} + \frac{3x^2}{8}$$

Then

$$(1+x)(1-x)^{-1/2}$$

$$\approx (1+x)\left[1 + \frac{x}{2} + \frac{3x^2}{8}\right]$$

$$\approx 1 + \frac{x}{2} + \frac{3x^2}{8} + x + \frac{x^2}{2} + \frac{3x^3}{8}$$

ignore this x^3 term

Answer $1 + \frac{3x}{2} + \frac{7x^2}{8}$

Q5

(i) Express $\dfrac{x-2}{(x+2)(x+1)} \equiv \dfrac{A}{(x+2)} + \dfrac{B}{(x+1)}$

$\dfrac{x-2}{(x+2)(x+1)}$

in partial fractions. $\dfrac{x-2}{(\quad X \quad)} \equiv \dfrac{A(x+1)}{(\quad X \quad)} + \dfrac{B(x+2)}{(\quad X \quad)}$ [6]

(ii) Hence expand

$x-2 = A(x+1) + B(x+2)$

$\dfrac{x-2}{(x+2)(x+1)}$

gives $B=-3$

& $A=4$

in ascending powers of x, as far as the term in x^3, using the binomial theorem. [11]

(iii) Find the range of values of x for which the expansion is valid. [3]

$$\frac{x-2}{(x+2)(x+1)} \equiv \frac{4}{(x+2)} - \frac{3}{(x+1)}$$

Take each separately.

$$\frac{4}{(x+2)} = 4(x+2)^{-1} = 4(2+x)^{-1} = 4\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$$

$$= 4\left(\tfrac{1}{2}\right)\left[\left(1+\frac{x}{2}\right)^{-1}\right]$$

$$= 2\left[1+(-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{1\times2}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{1\times2\times3}\left(\frac{x}{2}\right)^3 + \dots\right.$$

$$= 2\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots\right]$$

$$\approx 2 - x + \frac{x^2}{2} - \frac{x^3}{4}$$

$$\frac{3}{(x+1)} = 3(1+x)^{-1} = 3\left[1+(-1)(x) + \frac{(-1)(-2)}{1\times2}(x^2) + \frac{(-1)(-2)(-3)}{1\times2\times3}(x^3) + \dots\right]$$

$$= 3\left[1 - x + x^2 - x^3 + \dots\right]$$

$$\approx 3 - 3x + 3x^2 - 3x^3$$

Now combine these

$$\frac{4}{(x+2)} - \frac{3}{(x+1)} \approx 2 - x + \frac{x^2}{2} - \frac{x^3}{4} - 3 + 3x - 3x^2 + 3x^3$$

$$\approx -1 + 2x - \frac{5}{2}x^2 + \frac{11}{4}x^3$$

c) Valid when highlighted $|\quad| < 1$

$\left|\frac{x}{2}\right| < 1$ & $|x| < 1$

$|x| < 2$ & $|x| < 1$

Combining

Valid $-1 < x < +1$

Q6

Use the binomial theorem to expand

$$(1 + x + x^2)^{-1}$$

in ascending powers of x up to and including the term in x^3 [6]

$(1+x+x^2)^{-1}$ $\quad X = x + x^2$

$$(1+X)^{-1} = 1 + (-1)(X) + \frac{(-1)(-2)}{1\times 2}(X^2) + \frac{(-1)(-2)(-3)}{1\times 2\times 3}X^3$$

$$= 1 - X + X^2 - X^3 + \dots$$

$$= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + \dots$$

$$= 1 - (x+x^2) + (x^2+2x^3+x^4) - (x^3+3x^4+3x^5+x^6)$$

$$= 1 - x - x^2 + x^2 + 2x^3 + \cancel{x^4} - x^3 \cancel{-3x^4} \cancel{-3x^5} \cancel{-x^6} \cancel{+ \dots}$$

$$\approx 1 - x + x^3$$

Answer $\underline{\underline{1 - x + x^3}}$