Differential Equations

separate the variables

Q1

Given the differential equation

$$\frac{dy}{dx} = \frac{3y}{x+1}$$

boundary conditions

and that $x = 1$ when $y = 16$, express y in terms of x help to get c [10]

$$\frac{dy}{dx} = \frac{3y}{x+1}$$

$$+c_1 = +c_2$$

$$= +\cancel{c_2 - c_1}$$

$$+C$$

$$\int \frac{dy}{y} = \int \frac{3\,dx}{x+1}$$

$$\int \frac{1}{y}\,dy = 3\int \frac{1}{x+1}\,dx$$

$$\boxed{\ln y = 3\ln(x+1) + c}$$

$$\ln 16 = 3\ln(1+1) + c$$

$$\ln 16 = 3\ln(2) + c$$

$$\ln(16) = \ln(8) + c$$

$$\ln(16) - \ln(8) = c$$

$$\ln(2) = c$$

$$\ln(y) = 3\ln(x+1) + \ln(2)$$

$$\ln(y) = \ln(x+1)^3 + \ln(2)$$

$$\ln(y) = \ln\left[2(x+1)^3\right]$$

$$y = 2(x+1)^3$$

Q2

Solve the differential equation

$$\frac{1}{2}\frac{dy}{dx} = \frac{x\sqrt{y}}{x-1}$$

given that $y = 4$ when $x = 2$ [9]

Think $\frac{2x + \cdots}{x + \cdots}$

$$\frac{1}{2}\frac{dy}{\sqrt{y}} = \frac{x}{x-1}\,dx$$

$$\frac{dy}{\sqrt{y}} = \frac{2x}{x-1}\,dx$$

$$\int y^{-1/2}\,dy = \int \frac{2x}{x-1}\,dx$$

TOP HEAVY

$\int \frac{1}{x}\,dx = \ln x$

$$\frac{2x}{x-1} \equiv 2 + \frac{A}{x-1}$$

$$\frac{2x}{(x-1)} \equiv \frac{2(x-1)}{(x-1)} + \frac{A}{(x-1)}$$

$$2x \equiv 2(x-1) + A$$

$x=1$ $\quad 2 = A$

$$\int y^{-1/2}\,dy = \int 2 + \frac{2}{(x-1)}\,dx$$

$$2y^{1/2} = 2x + 2\ln(x-1) + c$$

$$\sqrt{y} = x + \ln(x-1) + c$$

$$\sqrt{4} = 2 + \ln(1) + c$$

$$2 = 2 + c$$

$$0 = c$$

Final Answer $\sqrt{y} = x + \ln(x-1)$

$$y = [x + \ln(x-1)]^2$$

Q3

Solve the differential equation

$$\left(\sin^2\theta\right)\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{4}{x^2}$$

Boundaries

given that $x = 3$ when $\theta = \frac{\pi}{4}$ [7]

$$\int x^2\,dx = 4\int \frac{1}{\sin^2\theta}\,d\theta$$

$$\int x^2\,dx = 4\int \operatorname{cosec}^2\theta\,d\theta$$

Page 10 Formula Booklet

Diff $\longrightarrow$

$\cot\theta \longrightarrow -\operatorname{cosec}^2\theta$

$$\frac{x^3}{3} = 4(-\cot\theta) + c$$

$$\frac{3^3}{3} = 4\left(-\cot\frac{\pi}{4}\right) + c$$

$$9 = -\frac{4}{\tan\frac{\pi}{4}} + c$$

$$9 = -4 + c$$

$$13 = c$$

Final Answer

$$\frac{x^3}{3} = 4(-\cot\theta) + 13$$

$$\frac{x^3}{3} + \frac{4}{\tan\theta} = 13$$

Q4

The voltage V of the battery in a smoke alarm can be modelled by

Rate of change of volt with time

$$\frac{dV}{dt} = -kV$$

where t is the number of days after installation and k is a positive constant.
The initial voltage is 9 volts and after 30 days the voltage drops to 4.5 volts.

$t=30$

(i) Solve this differential equation. [7]

The smoke alarm does not function properly if the battery's voltage drops below 1.5 volts.

(ii) Find after how many days the battery should be changed. [2]

$$\frac{dV}{dt} = -kV$$

$$\int \frac{1}{V}\, dV = \int -k\, dt$$

$$\ln V = -kt + c$$

Initally it was 9 volts

$t=0$, $V=9$

$$\ln 9 = \cancel{-k(0)} + c$$

$$\ln V = -kt + \ln 9$$

$$\ln(4.5) = -30k + \ln 9$$

$$30k = \ln(9) - \ln(4.5)$$

$$30k = \ln 2$$

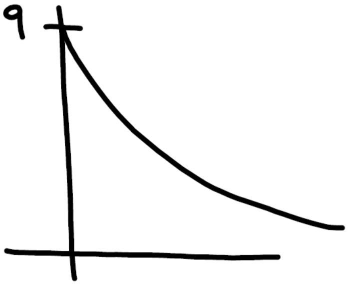
$$k = \frac{\ln 2}{30}$$

Final Answer $\ln V = -\frac{\ln 2}{30}t + \ln 9$

$$e^{V} = e^{-\frac{\ln 2}{30}t} \times e^{\ln 9}$$

$$V = 9e^{-\frac{\ln 2}{30}t}$$

Physics exp. decay



$$1.5 = 9e^{-\frac{\ln 2}{30}t}$$

etc

⋮

Ans 78 days.

Q5

In the atmosphere, the air pressure P (Pascals) decreases with the height h (km) above sea level at a rate that is proportional to the pressure.

(i) Model this by a differential equation. [2]

$$\frac{dP}{dh} \propto P$$

At sea level the air pressure is 100 000 Pa.
At 1 km above sea level the air pressure is 88 000 Pa.

(ii) By solving the differential equation, find the air pressure at 400 m above sea level. [8]

$$\frac{dP}{dh} \propto P$$

$$\frac{dP}{dh} = kP$$

$$\int \frac{1}{P}\,dP = \int k\,dh$$

$$\ln P = kh + c$$

when $h = 0 : P = 100000$

$$\ln 100000 = 0 + c \longrightarrow c = \ln 100000$$

$$\ln P = kh + \ln 100{,}000$$

when $h = 1 : P = 88000$

$$\ln 88000 = k + \ln 100000$$

$$k = -\ln \frac{100000}{88000}$$

$$k = -\ln\left(\frac{100}{88}\right)$$

Equation $\ln P = -\ln\left(\frac{100}{88}\right)h + \ln 100000$

$$P = 100000\,e^{-\ln\left(\frac{100}{88}\right)h}$$

check on calculator

(ii) when $h = 0.4$

$$P = 100000\,e^{-\ln\left(\frac{100}{88}\right)\times 0.4}$$

$$P = 95015$$

$$P = 95000 \quad \text{(3 sig. fig.)}$$

Q6

While on a skiing holiday in the Alps, Ben pours a cup of hot coffee from his flask.
The initial temperature of the coffee is 85 °C and the temperature of Ben's surroundings is a constant 10 °C.

$\frac{dx}{dt}$

The rate of change of the temperature x °C of the coffee after time t minutes ($t \geqslant 0$) is proportional to the difference in temperature between the coffee and the surroundings.

$\propto$ $(x-10)$

(i) Show that $\frac{dx}{dt} = -k(x - 10)$

where k is a positive constant. [3]

$$\frac{dx}{dt} \propto (x-10)$$

$$\frac{dx}{dt} = -k(x-10)$$

Negative because temperature is decreasing.

After two minutes, Ben's coffee has cooled to 65 °C.

(ii) Find the temperature of Ben's coffee after one **further** minute. [9]

(iii) Find t when $x = 35$ [3]

$$\frac{dx}{dt} = -k(x-10)$$

$$\int \frac{1}{(x-10)}\,dx = \int -k\,dt$$

$\ln(x-10) = -kt + c$ or $x = 10 + e^{-kt+c}$

Now we must use boundary conditions to get k & c

when $t = 0$ then $x = 85$

$\ln(85-10) = 0 + c$

$\ln(75) = c$ or $x = 10 + 75e^{-kt}$

when $t = 2$ the temperature is 65

$$65 = 10 + 75e^{-k(2)}$$

$$55 = 75e^{-2k}$$

$$\frac{55}{75} = e^{-2k}$$

$$\ln\left(\frac{55}{75}\right) = -2k$$

$$+0.155 = k$$

Equation

$$x = 10 + 75e^{-0.155t}$$

(ii) when $t = 3$ $x = 10 + 75e^{(-0.155)3} = 57.1$ 3 sig. fig.

(iii) find t when $x = 35$

$$35 = 10 + 75e^{-0.155t}$$

$$25 = 75e^{-0.155t}$$

$$\frac{25}{75} = e^{-0.155t}$$

$$\ln\left(\frac{25}{75}\right) = -0.155t$$

$$\frac{\ln\left(\frac{22}{75}\right)}{-0.155} = t \qquad t = 7.09 \text{ mins}$$