Geometric Progressions

Q1

Geometric Series

$u_n = ar^{n-1}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

The first four terms of a geometric series are

$$\underset{a}{0.51},\ \underset{ar}{0.0051},\ \underset{ar^2}{0.000051},\ \underset{ar^3}{0.00000051},\ \ldots$$

(i) Find the common ratio. [2]

(ii) Hence by finding the sum to infinity of this series, express the recurring decimal

$$0.5151515151\ldots$$

as a fraction in its simplest form. [4]

$a = 0.51 \quad r = 0.01$ or $\frac{1}{100}$

$0.515151\ldots$

$= 0.51 + 0.0051 + 0.000051$

$a = 0.51 \quad r = 0.01$

$$S_\infty = \frac{a}{1-r} = \frac{0.51}{0.99} = \frac{51}{99}$$

Q2

$\times 1.05$

A country's population at the end of each year is 5% greater than at the start of that year.
Model the population to be increasing at a constant rate.

(i) Find an expression for the population after t years. [2]

(ii) Find how many years it will take for the population to increase by 50% [5]

Population = Inital Population $\times 1.05^{t}$

Then to increase by 50%

$$150 = 100 \times 1.05^{t}$$

$$1.5 = 1.05^{t}$$

$$\log 1.5 = \log 1.05^{t}$$

$$\log 1.5 = t(\log 1.05)$$

$$\frac{\log 1.5}{\log 1.05} = t$$

$$8.310 = t$$

Try $t = 8$ years $\quad 100 \times 1.05^{8} = 147.7$

$t = 9$ years $\quad 100 \times 1.05^{9} = 155.1$

Q3

The time intervals $t_1, t_2, t_3, \ldots$ between consecutive bounces of a ball, bouncing on the ground, form the terms of a geometric progression.

The time interval t_5 is an eighth of the time interval t_2

(i) Find the common ratio. [4]

The sum of the first 6 time intervals is 6.3 seconds.

(ii) Find t_1 [3]

n^{th} term $= ar^{n-1}$

$t_1 = a$

$t_2 = ar$

$t_5 = ar^4$ $\qquad t_2 = ar$

$t_5 = \frac{1}{8} t_2$

$ar^4 = \frac{1}{8} ar$

$r^3 = \frac{1}{8}$ $\longrightarrow$ Common Ratio $= \frac{1}{2}$

$S_6 = 6{\cdot}3 = \dfrac{a(1-r^n)}{1-r}$

$6{\cdot}3 = \dfrac{a\left(1-\frac{1}{2}^6\right)}{1-\frac{1}{2}}$

$6{\cdot}3 = \dfrac{a\left(1-\frac{1}{64}\right)}{\frac{1}{2}}$

$\dfrac{63}{20} = a\left(\dfrac{63}{64}\right)$

$\dfrac{64}{20} = a$

check this!

$a = \dfrac{64}{20}$ $\qquad r = \dfrac{1}{2}$

$S_6 = \dfrac{\frac{64}{20}\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = \dfrac{\frac{64}{20}\left(\frac{63}{64}\right)}{\frac{1}{2}} = 6{\cdot}3$ ✓

Q4

(i) A sequence is defined recursively by

$$u_{n+1} = \frac{2}{3}u_n \quad \text{where } u_1 = 1$$

Find u_2, u_3 and u_4 [3]

(ii) State whether this sequence is convergent or divergent. [1]

A geometric series is formed by adding the terms of the sequence to give

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots$$

(iii) Find the common ratio of this geometric series. [1]

(iv) Find the sum to infinity of this geometric series. [2]

$u_1 = 1 \quad u_2 = \frac{2}{3} \quad u_3 = \frac{4}{9} \quad u_4 = \frac{6}{27}$

Converges because multiples by $2/3$ so gradually getting smaller

$1 + \frac{2}{3} + \frac{4}{9} + \frac{6}{27}$

$a = 1 \quad r = 2/3$ Ratio $= \frac{2}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

You can check on new calculator

$$\sum_{i=0}^{100} 1\left(\frac{2}{3}\right)^i$$

Q5

A sequence is defined by

$$u_{n+1} = 2bu_n \quad u_1 = 6 \quad (n = 1, 2, 3, 4 \ldots)$$

(i) Find u_2 and u_3 in terms of b [3]

(ii) Find the range of values of b for which the sequence converges. [3]

$u_1 = 6$ $\quad u_2 = 12b$ $\quad u_3 = 24b$

Range for converges

when $|2b| < 1$

$b < \frac{1}{2}$

Range of values $\quad -\frac{1}{2} < b < +\frac{1}{2}$