# Implicit Differentiation

term by term

## Q1

A curve has the equation

Think $4x \to 4$

$$4y - x = xy$$

(i) Show that $\boxed{\dfrac{dy}{dx} = \dfrac{y+1}{4-x}}$ [5]

(ii) Find the equation of the tangent to the curve at the point (3, 3). [3]

product

$$4y - x = \underset{u \cdot v}{xy}$$

$$4\frac{dy}{dx} - 1 = y \cdot 1 + x\frac{dy}{dx}$$

$$4\frac{dy}{dx} - x\frac{dy}{dx} = y + 1$$

$$(4-x)\frac{dy}{dx} = y + 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+1}{4-x}$$

$$y = mx + c$$

$$m = \text{gradient} = \frac{y+1}{4-x} = \frac{3+1}{4-3} = \frac{4}{1} = 4$$

$$y = 4x + c$$

Substitute $x=3$ & $y=3$ to get c

$$3 = 12 + c$$

$$-9 = c$$

Ans $y = 4x - 9$

**Q2**

A curve is defined by

Think $2x^2 \to 4x$

$$3x^2 + xy - 2y^2 = 0$$

$$3x^2 + \underbrace{xy}_{u \cdot v}^{\text{product}} - 2y^2 = 0$$

$$6x + y \cdot 1 + x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$$

$$6x + y = 4y\frac{dy}{dx} - x\frac{dy}{dx}$$

**(i)** Show that

$$\text{Gradient} = \frac{dy}{dx} = \frac{y + 6x}{4y - x}$$

$$6x + y = (\quad)\frac{dy}{dx}$$ [6]

**(ii)** Find the equation of the normal to the curve at the point (2, 3). [4]

Gradient tangent at (2,3) $\quad \text{Gradient} = \dfrac{3 + 6(2)}{12 - 2} = \dfrac{15}{10} = \dfrac{3}{2}$

Gradient Normal $= -\dfrac{2}{3}$

$y = mx + c$

$y = -\dfrac{2}{3}x + c$

Put in point (2,3)

$3 = -\dfrac{2}{3}(2) + c$

$3 = -\dfrac{4}{3} + c$

$\dfrac{13}{3} = c$

Ans $y = -\dfrac{2}{3}x + 4\tfrac{1}{3}$

$3y = -2x + 13$

## Q3

Find the equation of the normal to the curve

$$x^2 - 4xy + y^2 = 13$$

at the point (2, –1). [9]

Think
$x^2 \rightarrow 2x$
$y^2 \rightarrow 2y\frac{dy}{dx}$

$$x^2 - \underbrace{4x}_{u}\cdot\underbrace{y}_{v} + y^2 = 13$$

$$2x - \left[y.4 + 4x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$

$$2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x - 4y = (4x + 2y)\frac{dy}{dx}$$

$$\frac{2x-4y}{4x+2y} = \frac{dy}{dx}$$

Gradient of tangent at $(2,-1)$

$$\frac{2(2)-4(-1)}{4(2)+2(-1)} = \frac{4+4}{6} = \frac{8}{6} = \frac{4}{3}$$

Gradient of normal $= -3/4$

$$y = -3/4\, x + c$$

Substitute in $(2,-1)$ to get c

$$-1 = -3/4\,(2) + c$$

$$-1 = -6/4 + c$$

$$1/2 = c$$

$$y = -3/4\, x + 1/2$$

# Q4

$2\sin x - x = \tan y$

is shown in **Fig. 1** below.

Think $\tan^2 x \rightarrow \sec x$

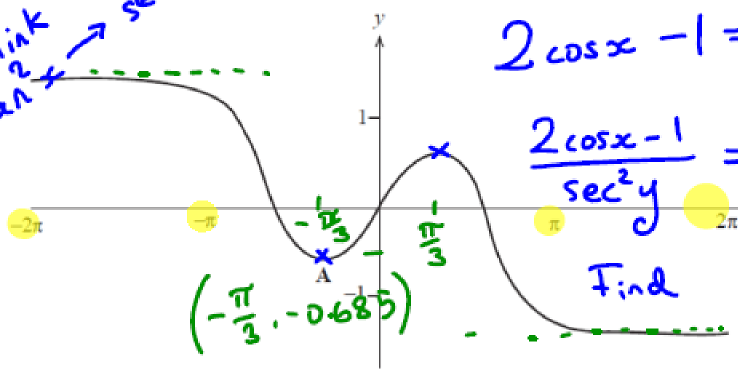



**Fig. 1**

$2\sin x - x = \tan y$

$2\cos x - 1 = (\sec^2 y)\frac{dy}{dx}$

$\frac{2\cos x - 1}{\sec^2 y} = \frac{dy}{dx}$

Find $\frac{2\cos x - 1}{\sec^2 y} = 0$

$2\cos x - 1 = 0$

(i) Find $\frac{dy}{dx}$ implicit ✓ [4]

(ii) Hence find the coordinates of the turning point labelled **A** in **Fig. 1** above. [5]

$\frac{dy}{dx} = 0$ Gradient $= 0$

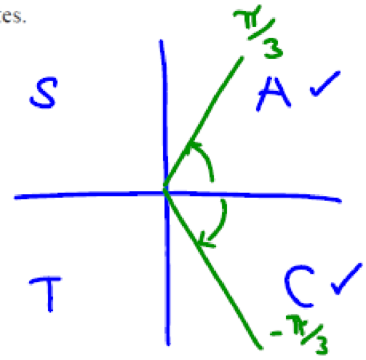
(iii) State the equations of the 2 horizontal asymptotes. [2]

Find $\frac{dy}{dx} = 0$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2})$




Find y co-ordinate by putting in $x = -\pi/3$

$2\sin x - x = \tan y$

$2\sin(-\frac{\pi}{3}) + \frac{\pi}{3} = \tan y$

$y = -0.685$

(iii) 2 horiz. asymptotes

Put in $x = 2\pi$

$2\sin(2\pi) - 2\pi = \tan(y)$

$0 - 2\pi = \tan(y)$

$\tan^{-1}(-2\pi) = y$

$-1.4129 = y$

Put in $x = -2\pi$

$2\sin(-2\pi) + 2\pi = \tan(y)$

$0 + 2\pi = \tan(y)$

$\tan^{-1}(2\pi) = y$

$1.4129 = y$

**Answer $y = -1.41$ and $y = 1.41$ (3 sig fig)**

# Q5

A curve has the equation

$$y\mathrm{e}^{-2x} = 2x + y^2$$

**(i)** Show that the gradient function of this curve is given by

$$\frac{2 + 2y\mathrm{e}^{-2x}}{\mathrm{e}^{-2x} - 2y}$$ [7]

The point P (0,1) lies on this curve.

**(ii)** Find the equation of the normal to this curve at the point P.
Give your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers. [4]

$$ye^{-2x} = 2x + y^2$$

product

$u \cdot v$

$$e^{-2x} \cdot \frac{dy}{dx} + y \cdot (-2e^{-2x}) = 2 + 2y\frac{dy}{dx}$$

$$e^{-2x}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2 - 2ye^{-2x}$$

$$(e^{-2x} - 2y)\frac{dy}{dx} = 2 - 2ye^{-2x}$$

$$\frac{dy}{dx} = \frac{2 - 2ye^{-2x}}{e^{-2x} - 2y}$$

Point (0,1)

$$m = \frac{dy}{dx} = \frac{2 + 2(1)(1)}{1 - 2(1)} = \frac{4}{-1} = -4$$

Gradient of Normal is $\frac{1}{4}$

Put $y = \frac{1}{4}x + c$ in (0,1) to get c value

$$1 = \frac{1}{4}(0) + c$$

$$1 = c$$

Answer $y = \frac{1}{4}x + 1$

$4y = x + 4$

Answer $0 = x - 4y + 4$

# Q6

**(i)** Differentiate

$$x^3 - 3x^2y + 2y^2 = 3$$

implicitly with respect to $x$. [5]

**(ii)** Hence find the equation of the tangent to the curve

$$x^3 - 3x^2y + 2y^2 = 3$$

at the point (1, 2). [3]

$$x^3 - 3x^2y + 2y^2 = 3$$

product $u \cdot v$

Think $2x^2 \to 4x$

$2y^2 \to 4y\frac{dy}{dx}$

$$3x^2 - \left[y \cdot 6x + 3x^2\frac{dy}{dx}\right] + 4y\frac{dy}{dx} = 0$$

$$3x^2 - 6xy - 3x^2\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$3x^2 - 6xy = (3x^2 - 4y)\frac{dy}{dx}$$

$$\frac{3x^2 - 6xy}{3x^2 - 4y} = \frac{dy}{dx}$$

At point $(1,2)$ $(x,y)$

$$\text{Gradient} = \frac{3(1)^2 - 6(1)(2)}{3(1)^2 - 4(2)} = \frac{3-12}{3-8} = \frac{-9}{-5} = \frac{9}{5}$$

Equation $y = mx + c$

$$y = \frac{9}{5}x + c$$

Put in $(1,2)$ to get $c$ value.

$$y = \frac{9}{5}x + \frac{1}{5}$$