All integration Part 1

Q1 Product Integration by Parts

(a) Find

$$\int xe^x \, dx \qquad [5]$$

(b) (i) Write

$$\frac{11-4x}{(x-2)(2x-1)}$$

in partial fractions. [5]

(ii) Hence find

$$\int \frac{11-4x}{(x-2)(2x-1)} \, dx \qquad [4]$$

$$\int \underset{u}{x}\, \underset{\frac{dv}{dx}}{e^x} dx = x \cdot e^x - \int e^x \cdot 1 \, dx$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$= x \cdot e^x - \int e^x dx$$

$$= x \cdot e^x - e^x + c \quad \checkmark$$

$$\boxed{u = x \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 1 \quad v = e^x}$$

(i) Write

$$\frac{11-4x}{(x-2)(2x-1)} \equiv \frac{A}{(x-2)} + \frac{B}{(2x-1)}$$

in partial fractions.

$$\frac{11-4x}{(\quad)(\quad)} = \frac{A(2x-1)}{(x-2)(2x-1)} + \frac{B(x-2)}{(x-2)(2x-1)}$$

$$11 - 4x \equiv A(2x-1) + B(x-2)$$

(ii) Hence find

$$\int \frac{11-4x}{(x-2)(2x-1)} \, dx$$

Pick $x = 2$ $\quad 3 = 3A \quad A \to 1$

Look x $\quad -4 = 2A + B \quad B \to -6$

$$\int \frac{11-4x}{(x-2)(2x-1)} \equiv \int \frac{1}{(x-2)} - \frac{6}{(2x-1)} \, dx$$

Remember

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int \frac{1}{2x} dx = \frac{1}{2}\ln|2x|$$

$$= \ln|x-2| - 6 \times \frac{1}{2} \ln|2x-1| + c$$

$$= \ln|x-2| - 3\ln|2x-1| + c$$

$$= \ln\left|\frac{x-2}{(2x-1)^3}\right| + c$$

Q2

Q2

Use the substitution $u = x - 4$ to find

$$\frac{du}{dx} = 1 \qquad du = dx$$

$$\int 5x(x-4)^3\,dx \qquad [6]$$

See Brackets Think Substitution

$$\int 5x \cdot u^3 \cdot dx$$

$$= \int 5(u+4) \cdot u^3 \cdot dx$$

$$= \int 5(u+4) \cdot u^3 \cdot du$$

Everything has changed from x to u

$$= \int 5u^4 + 20u^3\,du$$

$$= \frac{5u^5}{5} + \frac{20u^4}{4}$$

$$= u^5 + 5u^4 + c$$

$$= (x-4)^5 + 5(x-4)^4 + c$$

Job done!

Find the **exact** value of

By Parts

$$\int_1^2 x^2 \ln x\,dx = \qquad [7]$$

The formula

$$\int u \cdot \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

~~$u = x^2$ $\frac{dv}{dx} = \ln x$ $\frac{du}{dx} = 2x$ $v =$ can't do this~~

$$u = \ln x \qquad \frac{dv}{dx} = x^2$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{3}x^3$$

$$\int_1^2 x^2 \ln x\,dx = (\ln x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \left[\frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3}\right]^2 - \left[\frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3}\right]_1$$

$$= \left[\frac{8\ln 2}{3} - \frac{8}{9}\right] - \left[\cancel{\frac{\ln 1}{3}} - \frac{1}{9}\right]$$

$$= \frac{8\ln 2}{3} - \frac{7}{9}$$

Q3

(i) Write $\dfrac{3x+4}{x(x+1)}$ in partial fractions. [6]

(ii) Hence find the exact area bounded by the curve $y = \dfrac{3x+4}{x(x+1)}$, the x-axis and the lines $x = 2$ and $x = 3$ [7]

[The curve does not cross the x-axis between 2 and 3]

$$\int_{x=2}^{x=3}$$

$$\frac{3x+4}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$$

$$3x+4 = A(x+1) + Bx$$

Put in $x=0$ $\quad 4 = A$

Put in $x=-1$ $\quad 1 = -B \quad B = -1$

$$\frac{3x+4}{x(x+1)} = \frac{4}{x} - \frac{1}{x+1}$$

$$\int \frac{3x+4}{x(x+1)}\,dx = \int \frac{4}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= 4\ln x - \ln(x+1) + c$$

$$= \ln(x)^4 - \ln(x+1) + c$$

$$\int_2^3 \frac{3x+4}{x(x+1)}\,dx = \Big[\ln(x^4) - \ln(x+1)\Big]_2^3$$

$$= \Big[\ln(81) - \ln(4)\Big] - \Big[\ln(16) - \ln(3)\Big]$$

$$= \ln(81 \times 3) - \ln(4 \times 16)$$

$$= \ln(243) - \ln(64)$$

$$= \ln\left(\frac{243}{64}\right)$$

Q4

Product

Find

$$\int x \operatorname{cosec}^2 x \, dx = -x \cot x - \int -\cot x \cdot 1 \, dx$$

[7]

$$= -x \cot x + \int \cot x \, dx$$

$$= -x \cot x + \ln|\sin x| + c$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$u = x$ $\quad \frac{dv}{dx} = \operatorname{cosec}^2 x$

$\frac{du}{dx} = 1$ $\quad v = -\cot x$

Q5

Q5 Product By Parts

(a) Find $\int (2x^4)(\ln 3x)\,dx$ [6]

(b) Use partial fractions to find

$$\int \frac{x+9}{3-2x-x^2}\,dx$$ [8]

$$\frac{x+9}{(3+x)(1-x)}$$

$$\frac{x+9}{(3+x)(1-x)} = \frac{A}{(3+x)} + \frac{B}{(1-x)}$$

$$x+9 = A(1-x) + B(3+x)$$

$x=1$ $\quad 10 = 4B \longrightarrow B = 5/2$

$x=-3$ $\quad 6 = 4A \longrightarrow A = 3/2$

$$\int 2x^4 \cdot \ln 3x\,dx$$

$\cancel{u = 2x^4}$ · $\cancel{\frac{dv}{dx} = \ln(3x)}$

$u = \ln(3x)$ $\quad \frac{dv}{dx} = 2x^4$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{2x^5}{5}$

$$\int (\ln 3x)(2x^4)\,dx = \frac{2x^6}{6}\cdot \ln(3x) - \int \frac{2}{6}x^6 \cdot \frac{1}{x}\,dx$$

$$= \frac{x^6 \cdot \ln(3x)}{3} - \int \frac{1}{3}x^5\,dx$$

$$= \frac{x^6 \cdot \ln(3x)}{3} - \frac{1}{18}x^6 + c$$

$$\frac{x+9}{(3+x)(1-x)}$$

$$\frac{x+9}{(3+x)(1-x)} = \frac{A}{(3+x)} + \frac{B}{(1-x)}$$

$$x+9 = A(1-x) + B(3+x)$$

$x=1$ $\quad 10 = 4B \longrightarrow B = 5/2$

$x=-3$ $\quad 6 = 4A \longrightarrow A = 3/2$

$$\int \frac{x+9}{(3+x)(1-x)}\,dx = \int \frac{3/2}{(3+x)} + \frac{5/2}{(1-x)}\,dx$$

$$= \frac{3}{2}\ln(3+x) - 5/2\,\ln(1-x)$$

$$= \ln\sqrt{(3+x)^3} - \ln\sqrt{(1-x)^5}$$

$$= \ln\left(\frac{\sqrt{(3+x)^3}}{\sqrt{(1-x)^5}}\right) + c$$

Q6

A wooden plinth for a statue can be modelled by the volume generated when the area bounded by the curve $y = \sec x$, the x-axis, the y-axis and the line $x = a$ $\left(0 < a < \frac{\pi}{2}\right)$ is rotated through 2π radians about the x-axis.

(a) Find an expression for V, the volume of the plinth. [5]

(b) Find V if $a = \frac{\pi}{4}$ [1]

$$\text{Volume} = \pi \int y^2 \, dx$$

$$\text{Volume} = \pi \int_0^a \sec^2 x \, dx$$

$$\text{Volume} = \left[\pi \tan x\right]_0^a$$

$$\text{Volume} = \left[\pi \tan x\right]^a - \left[\pi \tan x\right]_0$$

$$\text{Volume} = \pi \tan a - 0$$

$$\text{Volume} = \pi \tan a$$

(ii) If $a = \pi/4$

$$\text{Volume} = \pi \cdot \tan(\pi/4)$$

$$= \pi \cdot 1$$

$$\text{Volume} = \pi$$

Q7

Find

(i) $\int_0^2 x\mathrm{e}^{-x}\,\mathrm{d}x$ [7]

(ii) $\int \sin^3 x\ \mathrm{d}x$ [7]