Integration part 2

Q1

Using the substitution $x = \sqrt{2}\sin\theta$, or otherwise, show that

$$\int_0^1 \sqrt{2 - x^2}\,\mathrm{d}x = \frac{2+\pi}{4}$$ [12]

$x = \sqrt{2}\sin\theta$

$\frac{dx}{d\theta} = \sqrt{2}\cos\theta$

$dx = \sqrt{2}\cos\theta\, d\theta$

$$\int \sqrt{2 - x^2}\, dx$$

$$= \int \sqrt{2 - 2\sin^2\theta} \cdot \sqrt{2}\cos\theta\, d\theta$$

$$= \int \sqrt{2\cos^2\theta} \cdot \sqrt{2}\cos\theta\, d\theta$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$$

$$\boxed{\cos 2\theta + 1 = 2\cos^2\theta}$$

$$= \int \sqrt{2}\cos\theta \cdot \sqrt{2}\cos\theta\, d\theta$$

$$= \int 2\cos^2\theta\, d\theta$$

$$= \int (\cos 2\theta + 1)\, d\theta$$

$$= \frac{1}{2}\sin 2\theta + \theta$$

$1 = \sqrt{2}\sin\theta$

$1/\sqrt{2} = \sin\theta$

$\pi/4 = \theta$

$$\int_{x=0}^{x=1} \sqrt{2 - x^2}\, dx = \left[\frac{1}{2}\sin 2\theta + \theta\right]_{x=0,\ \theta=0}^{x=1,\ \theta=\pi/4}$$

$x = \sqrt{2}\sin\theta$

$0 = \sqrt{2}\sin\theta$

$0 = \theta$

$$= \left[\frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}\right] - \left[\frac{1}{2}\sin 0 + 0\right]$$

$$= \frac{1}{2}\cdot 1 + \frac{\pi}{4}$$

$$= \frac{1}{2} + \frac{\pi}{4}$$

$$= \frac{2+\pi}{4}$$

Q2

The designers of a bowl use an area rotated through 2π radians around the x-axis as the basis for their design.
The area used is between the curve $y = 4\sqrt{x+2}$, the x-axis and the lines $x = 0$ and $x = a$, as shown in **Fig. 1** below.

Volume $= \pi \int y^2 \, dx$

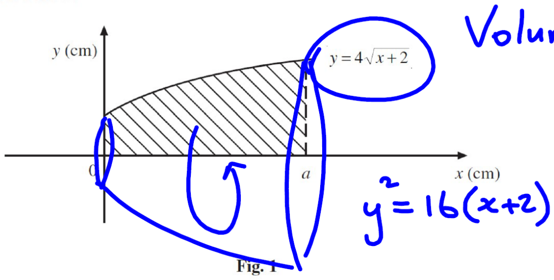

Fig. 1

(i) Find an expression for the capacity of the bowl in terms of a. [7]

The specification requires the capacity of the bowl to be 1000 cm^3

(ii) Find the value of a correct to one decimal place. [7]

$$\text{Volume} = \int_0^a \pi(16)(x+2)\,dx = 16\pi \int_0^a (x+2)\,dx$$

$$\text{Volume} = \left[16\pi\left(\frac{x^2}{2} + 2x\right)\right]_0^a$$

$$= \left[16\pi\left(\frac{a^2}{2} + 2a\right)\right]^a - \cancel{\left[16\pi(0)\right]_0}$$

$$\text{Volume} = 16\pi\left(\frac{a^2}{2} + 2a\right)$$

$$= 8\pi a^2 + 32\pi a$$

(ii) Volume $= 1000$ cm^3

$$8\pi a^2 + 32\pi a = 1000$$

$$8\pi a^2 + 32\pi a - 1000 = 0$$

EQUATION SOLVER

$\boxed{a = 4.617}$ & ~~$a = -8.617$~~ NOT possible

check

$$\int_0^{4.617} \pi \times 16 \times (x+2)\,dx$$ on CALCULATOR

Q3

(a) Find

$$\int (16x^3)(\ln x)\,dx \qquad [7]$$

(b) Find

$$\int (\sin x + \cos x)^2\,dx \qquad [6]$$

By Parts

$$\int 16x^3 \cdot \ln x\,dx$$

$$\cancel{u = 16x^3 \quad \frac{dv}{dx} = \ln x}$$

$$\cancel{\frac{du}{dx} = 48x^3 \quad v \; \times}$$

$u = \ln x \qquad \frac{dv}{dx} = 16x^3 \qquad v = 4x^4$

$\frac{du}{dx} = \frac{1}{x}$

$$\boxed{\int u \cdot \frac{dv}{dx} = uv - \int v \frac{du}{dx}\,dx}$$

$$= (\ln x)\,4x^4 - \int 4x^4 \cdot \frac{1}{x}\,dx$$

$$= (4x^4)\ln x - \int 4x^3\,dx$$

$$= (4x^4)\ln x - x^4 + c$$

$(\sin x + \cos x)(\sin x + \cos x)$

$$= \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$$

$$= \int 1 + 2\sin x\cos x\,dx$$

$$= \int 1 + \sin 2x\,dx$$

$$= x - \frac{1}{2}\cos 2x + c$$

Think DC Comics are very negative

Q4

(a) Using the substitution $u = 2 + x$ find

$u = 2 + x$

$$\int x(2+x)^{10}\,dx \qquad [7]$$

$\frac{du}{dx} = 1 \qquad du = dx$

(b) Find the **exact** value of

$$\int_0^{\frac{\pi}{4}} 8x\cos 2x\,dx \qquad [6]$$

By Parts

$u = 8x \qquad \frac{dv}{dx} = \cos 2x$

$$\int (u-2)\,u^{10}\,du$$

$$\int u^{11} - 2u^{10}\,du$$

$$= \frac{u^{12}}{12} - \frac{2u^{11}}{11} + c$$

$$= \frac{(x+2)^{12}}{12} - \frac{2(x+2)^{11}}{11} + c$$

Q5

(a) Find $\int \sin^3 x \, dx$ Tricky. Think $\boxed{\sin^2 x + \cos^2 x = 1}$ [5]

(b) Find the **exact** value of the volume generated when the area bounded by the curve $y = 2e^x$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated through 2π radians about the x-axis. [7]

$$\int \sin^3 x \, dx = \int \sin x (1 - \cos^2 x) \, dx$$

$$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$$

Substitution

let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\int \sin x \, dx = -\cos x$$

$$\int \sin x \cos^2 x \, dx = \int -u^2 \, du = -\frac{u^3}{3} = -\frac{\cos^3 x}{3}$$

Combine the 2 parts

Ans $\quad -\cos x + \dfrac{\cos^3 x}{3} + c$

$$\text{Volume} = \pi \int y^2 \, dx$$

$$= \pi \int_1^4 (2e^x)^2 \, dx$$

$$= 4\pi \int_1^4 e^{2x} \, dx$$

$$= \left[\frac{4\pi e^{2x}}{2} \right]_1^4$$

$$= [2\pi e^8] - [2\pi e^2]$$

$$= 2\pi (e^8 - e^2)$$

You can check $\int_\square^\square \square \, dx$ button

Q6

Volume $= \pi \int y^2 \, dx$

A bowl is formed by rotating through 2π radians about the x-axis, the arc of the curve

$$y = \sqrt{5x}$$

between $x = 0$ and $x = a$, where a is a positive constant.
The bowl is full of water.
Find the volume of water in the bowl. [6]

$$\text{Volume} = \int_0^a \pi \left(\sqrt{5x}\right)^2 dx$$

$$\text{Volume} = \pi \int_0^a 5x \, dx$$

$$\text{Volume} = \pi \left[\frac{5x^2}{2}\right]^a - \pi \left[\frac{5x^2}{2}\right]_0$$

$$\text{Volume} = \pi \frac{5a^2}{2} - 0$$

$$\text{Volume} = \frac{5\pi a^2}{2}$$ Answer

Q7

Because of the ( )

Using the substitution $u = 1 + x$, find the **exact** value of

$$\int_{-1}^{0} x(1+x)^{\frac{1}{2}}\,\mathrm{d}x$$

[8]

$u = 1 + x$

$\frac{du}{dx} = 1$

$du = dx$

$u = 1 + x$

$u - 1 = x$

Change everything into u

$$\int_{x=-1}^{x=0} (u-1)\,u^{1/2}\,du$$

$$= \int_{x=-1}^{x=0} u^{3/2} - u^{1/2}\,du$$

$$= \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_{x=-1}^{x=0}$$

$$= \left[\frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3}\right]_{x=-1}^{x=0}$$

$$= \left[\frac{2 \times 1^{5/2}}{5} - \frac{2 \times 1^{3/2}}{3}\right] - \left[0 - 0\right]$$

$$= \frac{2}{5} - \frac{2}{3}$$

$$= \frac{6}{15} - \frac{10}{15}$$

$$= -\frac{4}{15}$$

Ans $-\frac{4}{15}$