Partial fractions the Integration

Q1

(b) (i) Write

$$\frac{11-4x}{(x-2)(2x-1)}$$

in partial fractions. [5]

(ii) Hence find

$$\int \frac{11-4x}{(x-2)(2x-1)}\,dx$$ [4]

(i) $$\frac{11-4x}{(x-2)(2x-1)} \equiv \frac{A}{(x-2)} + \frac{B}{(2x-1)}$$

$$\frac{11-4x}{(\quad)(\quad)} \equiv \frac{A(2x-1)}{(x-2)(2x-1)} + \frac{B(x-2)}{(\quad)(\quad)}$$

$$11-4x = A(2x-1) + B(x-2)$$

Put in $x=2$ $\quad 3 = 3A \longrightarrow A=1$

Look at x terms $\quad -4 = 2A + B$

$$-4 = 2\times 1 + B \longrightarrow B=6$$

$$-4 = 2 + B$$

So now,

$$\int \frac{11-4x}{(x-2)(2x-1)}\,dx = \int \frac{1}{(x-2)} + \frac{6}{2x-1}\,dx$$

These make me think about the 9 special rules $\quad \int \frac{1}{x}\,dx = \ln|x$

$$= \ln|x-2| + \frac{6}{2}\ln|2x-1|$$

$$= \ln|x-2| + 3\ln|2x-1|$$

$$= \ln|x-2| + \ln|(2x-1)^3|$$

$$= \ln|(x-2)(2x-1)^3|$$

Q2

(i) Write $\dfrac{3x+4}{x(x+1)}$ in partial fractions. [6]

(ii) Hence find the exact area bounded by the curve $y = \dfrac{3x+4}{x(x+1)}$, the x-axis and the lines $x = 2$ and $x = 3$ [7]
[The curve does not cross the x-axis between 2 and 3]

$$\frac{3x+4}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{(x+1)}$$

$$\frac{3x+4}{x(x+1)} \equiv \frac{A(x+1)}{x(x+1)} + \frac{Bx}{x(x+1)}$$

$$3x+4 \equiv A(x+1) + B(x) \quad \text{gives} \quad A=4, \; B=-1$$

$$\int_2^3 \frac{3x+4}{x(x+1)}\,dx = \int_2^3 \frac{4}{x} - \frac{1}{x+1}\,dx$$

$$= \Big[4\ln|x| - \ln|x+1|\Big]_2^3$$

$$= \Big[4\ln|3| - \ln|4|\Big] - \Big[4\ln|2| - \ln|3|\Big]$$

$$= \ln(3^4) - \ln|4| - \ln(2^4) + \ln|3|$$

$$= \ln\left(\frac{3^4 \times 3}{4 \times 2^4}\right)$$

$$= \ln\left(\frac{3^5}{2^6}\right) = \ln\left|\frac{243}{32}\right|$$

Q3

$$\int \frac{x+9}{3-2x-x^2}\,dx$$ [8]

must factorise

$3-2x-x^2$

$-[x^2+2x-3]$

$-[(x+3)(x-1)]$

$=(x+3)(1-x)$

[8]

$$\frac{x+9}{(x+3)(1-x)}$$

$$\equiv \frac{A}{(x+3)} + \frac{B}{(1-x)}$$

$$\frac{x+9}{(x+3)(1-x)} \equiv \frac{A(1-x)}{(\ )(\ )} + \frac{B(x+3)}{(\ )(\ )}$$

$$x+9 = A(1-x) + B(x+3)$$

Put in $x=1$

$10 = 4B \qquad B = 2.5 = \frac{5}{2}$

Put in $x=-3$

$6 = 4A \qquad A = 1.5 = \frac{3}{2}$

$$\int \frac{x+9}{3-2x-x^2}\,dx = \int \frac{\frac{3}{2}}{(x+3)} + \frac{\frac{5}{2}}{(1-x)}\,dx$$

$$= \frac{1}{2}\int \frac{3}{(x+3)} + \frac{5}{(1-x)}\,dx$$

$$= \frac{1}{2}\left(3\ln|x+3| + \frac{5}{-1}\ln|1-x|\right) + c$$

$$= \frac{3}{2}\ln|x+3| - \frac{5}{2}\ln|1-x| + c$$

maybe combine these altogether using rules of logs

$$= \frac{1}{2}\left[\ln(x+3)^3 - \ln(1-x)^5\right] + c$$

$$= \frac{1}{2}\ln\left(\frac{(x+3)^3}{(1-x)^5}\right) + c$$

$$= \ln\left(\sqrt{\frac{(x+3)^3}{(1-x)^5}}\right) + c$$

Q4

Show that $\int_1^2 \frac{1}{x^2(x+1)}\,dx = \frac{1}{2} + \ln\left(\frac{3}{4}\right)$.

Partial Fractions

$$\frac{1}{x^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)}$$

Repeated Factor.

$$\frac{1}{x^2(x+1)} \equiv \frac{Ax(x+1)}{x^2(x+1)} + \frac{B(x+1)}{x^2(x+1)} + \frac{Cx^2}{x^2(x+1)}$$

$$1 \equiv Ax(x+1) + B(x+1) + Cx^2$$

$x=0$ $\quad 1 = B \qquad B = 1$

$x=-1$ $\quad 1 = C \qquad C = 1$

Equate x^2 $\quad 0 = A + C \longrightarrow A = -1$

$$\int_1^2 \frac{1}{x^2(x+1)}\,dx = \int_1^2 \frac{-1}{x} + x^{-2} + \frac{1}{x+1}\,dx$$

$$= \left[-\ln|x| - \frac{1}{x} + \ln(x+1)\right]_1^2$$

$$= \left[-\ln(2) - \frac{1}{2} + \ln 3\right] - \left[-\ln(1) - 1 + \ln 2\right]$$

$$= -\ln 2 - \frac{1}{2} + \ln 3 + \cancel{\ln(1)} + 1 - \ln 2$$

$$= -\frac{1}{2} + 1 + \ln 3 - \ln 2 - \ln 2$$

$$= \frac{1}{2} + \ln\left(\frac{3}{2\times 2}\right)$$

$$= \frac{1}{2} + \ln\left(\frac{3}{4}\right)$$

Q5

Evaluate $\int_0^1 \frac{2x+3}{x^2+3x+2}\,dx$ exactly, expressing your answer as a single logarithm.

Factorise

x^2+3x+2

$(x+2)(x+1)$

Partial Fractions

$$\frac{2x+3}{(x+2)(x+1)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$2x+3 \equiv A(x+1) + B(x+2)$$

Put in $x=-1$ $\quad 1 = B$

Put in $x=-2$ $\quad -1 = -A \quad A=1$

$$\int_0^1 \frac{2x+3}{x^2+3x+2}\,dx = \int_0^1 \frac{1}{(x+2)} + \frac{1}{(x+1)}\,dx$$

$$= \Big[\ln|x+2| + \ln|x+1|\Big]_0^1$$

$$= \Big[\ln|3| + \ln|2|\Big] - \Big[\ln|2| + \ln|1|\Big]$$

$$= \ln|6| - \ln|2|$$

$$= \ln\left|\frac{6}{2}\right|$$

$$= \ln|3|$$

Q6

$$\int \frac{2x^2+x-7}{(x+3)(x-1)}\,dx$$

Top Heavy Fraction $\frac{2x^2+\cdots}{x^2+\cdots}$

$$\frac{2x^2+x-7}{(x+3)(x-1)} \equiv 2 + \frac{A}{(x+3)} + \frac{B}{(x-1)}$$

$$2x^2+x-7 \equiv 2(x+3)(x-1) + A(x-1) + B(x+3)$$

Put in $x=1$ $\quad -4 = 4B \quad B=-1$

Put in $x=-3$ $\quad 8 = -4A \quad A=-2$

$$\int \frac{2x^2+x-7}{(x+3)(x-1)}\,dx = \int 2 - \frac{2}{(x+3)} - \frac{1}{(x-1)}\,dx$$

$$= 2x - 2\ln|x+3| - \ln|x-1| + c$$

$$= 2x - \left[2\ln|x+3| + \ln|x-1|\right] + c$$

$$= 2x - \ln\left|(x+3)^2(x-1)\right| + c$$