Parametric differentiation

Q1

A curve has parametric equations

$$x = \sin^2 t, \quad y = \tan t, \quad -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}.$$

a Show that the tangent to the curve at the point where $t = \frac{\pi}{4}$ passes through the origin.

b Find a cartesian equation for the curve in the form $y^2 = \mathrm{f}(x)$.

$x = \sin^2 t$

$x = (\sin t)^2$

$\frac{dx}{dt} = 2\sin t \cos t$

$y = \tan t$

$\frac{dy}{dx} = \sec^2 t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \sec^2 t \times \frac{1}{2\sin t \cos t}$

$m = \frac{dy}{dx} = \frac{1}{2\sin t \cos^3 t}$

Gradient when $t = \pi/4$

$m = \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^3}$

$m = 2$

Point when $t = \pi/4$ $\left(\frac{1}{2}, 1\right)$

$y = 2x + c$

$1 = 2\left(\frac{1}{2}\right) + c$

$1 = 1 + c$

$0 = c$

Ans $y = \frac{1}{2}x + 0$

goes through origin

b) Cartesian that links $\sin^2 t$ & $\tan t$

$x = \sin^2 t$

$\tan t = y$

$\frac{\sin t}{\cos t} = y$

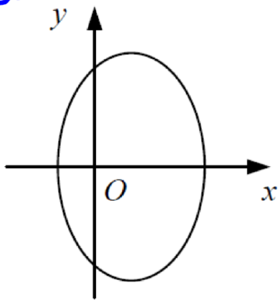
$\frac{\sin^2 t}{\cos^2 t} = y^2$

$\frac{x}{1-x} = y^2$ Ans $y^2 = \frac{x}{1-x}$

Q2

$x = 1 - 2\cos\theta$ $\qquad$ $y = 3\sin\theta$

$\frac{dx}{d\theta} = 2\sin\theta$ $\qquad$ $\frac{dy}{d\theta} = 3\cos\theta$



$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= 3\cos\theta \times \frac{1}{2\sin\theta}$$
$$= \frac{3\cos\theta}{2\sin\theta}$$

The diagram shows the ellipse with parametric equations

$$x = 1 - 2\cos\theta, \quad y = 3\sin\theta, \quad 0 \leq \theta < 2\pi.$$

a Find $\frac{dy}{dx}$ in terms of θ.

b Find the coordinates of the points where the tangent to the curve is

i parallel to the x-axis,

ii parallel to the y-axis.

(b)(i) parallel to x axis means gradient $= 0$ so find values of θ so that $\frac{dy}{dx} = 0$

$m = 0 = \frac{3\cos\theta}{2\sin\theta}$

solve $3\cos\theta = 0$

$\cos\theta = 0$

cos curve

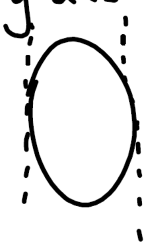
$\theta = \frac{\pi}{2}$ & $\frac{3\pi}{2}$

when $\theta = \frac{\pi}{2}$ $\qquad$ when $\theta = \frac{3\pi}{2}$

$(1, 3)$ $\qquad$ $(1, -3)$

which makes sense when you look up at diagram

(b)(ii) line parallel to y axis



when gradient is not defined so when $\frac{dy}{dx}$ doesn't exist

solve

$2\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0$ & π & $\cancel{2\pi}$ not interval

when $\theta = 0$ $\qquad$ when $\theta = \pi$

$(-1, 0)$ $\qquad$ $(3, 0)$

again, it makes sense if you look at the diagram

Q3 A curve has the parametric equations

$$x = \tan^2 t, \quad \boxed{y = \cos t,} \quad 0 < t < \frac{\pi}{2}$$

(x, y)

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. **(3)**

(b) Find an equation of the tangent to the curve where $t = \frac{\pi}{4}$ **(5)**

Point $\left(1, \frac{1}{\sqrt{2}}\right)$

(c) Find a cartesian equation for the curve. **(4)**

$$\frac{dx}{dt} = ? = \frac{dx}{du} \times \frac{du}{dt} = 2u \times \sec^2 t = 2\tan t \cdot \sec^2 t$$

$$x = (\tan t)^2$$

$$x = u^2 \qquad \boxed{u = \tan t}$$

$$\frac{dx}{du} = 2u \qquad \frac{du}{dt} = \sec^2 t$$

$$y = \cos t$$

$$\frac{dy}{dt} = -\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\sin t \times \frac{1}{2\tan t \sec^2 t} = -\frac{\cos^3 t}{2}$$

Gradient at point $t = \pi/4$

$$m = -\frac{\cos^3 \frac{\pi}{4}}{2}$$

$$m = \frac{-\left(\frac{1}{\sqrt{2}}\right)^3}{2}$$

$$m = -\frac{\sqrt{2}}{8}$$

$$y = mx + c$$

$$y = -\frac{\sqrt{2}}{8}x + c$$

Substitute in the point $\left(1, \frac{1}{\sqrt{2}}\right)$ to get

$$y = -\frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}$$

(iii) Cartesian

$$x = \tan^2 t \qquad y = \cos t$$

$$\tan^2 t + 1 = \sec^2 t \qquad y^2 = \cos^2 t$$

$$x + 1 = \frac{1}{y^2} \qquad \frac{1}{y^2} = \frac{1}{\cos^2 t}$$

$$y^2 = \frac{1}{1+x} \qquad \frac{1}{y^2} = \sec^2 t$$

Q4

A curve is defined by the parametric equations

$$x = 2t \qquad y = t^3 - 3t$$

Find in terms of t

(i) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3t^2 - 3 \times \frac{1}{2} = \frac{3t^2-3}{2}$ [4]

(ii) $\boxed{\frac{d^2y}{dx^2}}$ trickiest [3]

$x = 2t$

$\frac{dx}{dt} = 2$

$y = t^3 - 3t$

$\frac{dy}{dt} = 3t^2 - 3$

(ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{3t^2}{2} - \frac{3}{2}\right) \times \frac{1}{2}$

$= (3t) \times \frac{1}{2}$

$= \frac{3t}{2}$

Q5

A curve is given by the parametric equations

$$x = \sin\theta, \quad y = \sin 2\theta, \quad \boxed{0 \le \theta \le \tfrac{\pi}{2}.}$$

a Find the coordinates of any points where the curve meets the coordinate axes.

b Find an equation for the tangent to the curve that is parallel to the x-axis. $x = \frac{\pi}{4}$ $y = 1$

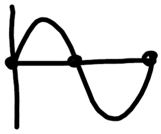
c Find a cartesian equation for the curve in the form $y = f(x)$.

$x = \sin\theta$

cut y axis when

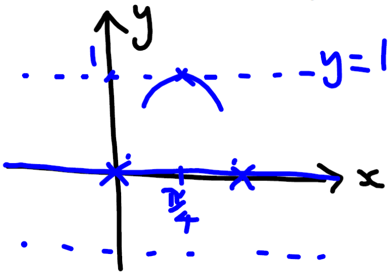
$x = 0$

$0 = \sin\theta$



$\theta = 0$

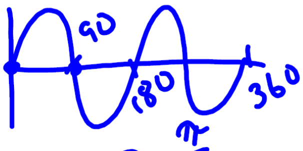
$(0,0)$



cut x axis when $y = 0$

$y = \sin 2\theta$

$0 = \sin 2\theta$



$\theta = 0, \frac{\pi}{2}$

$\frac{dy}{dx} = 0$ = Gradient = Horizontal $(0,0)$ $(1,0)$

$x = \sin\theta$ $\quad$ $y = \sin 2\theta$

$\frac{dx}{d\theta} = \cos\theta$ $\quad$ $\frac{dy}{d\theta} = 2\cos 2\theta$

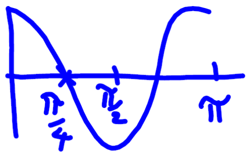
$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2\cos 2\theta \times \frac{1}{\cos\theta} = \frac{2\cos 2\theta}{\cos\theta}$$

when gradient

$2\cos 2\theta = 0$

$\cos 2\theta = 0$

$\cos\left(2 \times \frac{\pi}{4}\right) = 0$



$\frac{0}{\text{anything}} = 0$

$x = \sin\theta, \quad y = \sin 2\theta,$

oordinates of any points wl

Cartesian

$x = \sin\theta$ $\quad$ $y = 2\sin\theta\cos\theta$

$y = 2x \cdot \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$x^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - x^2$

$$\boxed{y = 2x \cdot \sqrt{1 - x^2}}$$

Q6

$$\frac{dy}{d\theta} = 2\sin 2\theta$$

A curve is defined parametrically by

$$\boxed{x = 2\sin\theta + 1} \qquad \boxed{y = 1 - \cos 2\theta}$$

DC comics are very dark

(i) Show that $\frac{dx}{d\theta} = 2\cos\theta$

$$\frac{dy}{dx} = 2\sin\theta$$ [5]

(ii) Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$ [5]

(iii) Find $\frac{d^2y}{dx^2}$ [3]

(i) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= 2\sin 2\theta \times \frac{1}{2\cos\theta} = \frac{\cancel{2} \times 2\sin\theta\cancel{\cos\theta}}{\cancel{2}\cancel{\cos\theta}}$

$\frac{dy}{dx} = 2\sin\theta$

(ii) $y = mx + c$

m: $\frac{dy}{dx}$

At $\theta = \frac{\pi}{4}$. $\frac{dy}{dx} = 2 \times \sin\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$y = \sqrt{2}x + c$

$1 = \sqrt{2}(\sqrt{2} + 1) + c$

$1 = 2 + \sqrt{2} + c$

$-1 - \sqrt{2} = c$

At $\theta = \frac{\pi}{4}$

$x = 2\sin\left(\frac{\pi}{4}\right) + 1$

$x = \sqrt{2} + 1$

$(\sqrt{2} + 1, 1)$

$y = 1 - \cos\left(2 \times \frac{\pi}{4}\right)$

$y = 1$

$y = \sqrt{2}x - 1 - \sqrt{2}$

gradient $\frac{dy}{dx}$

using point $\theta = \frac{\pi}{4}$